Thus, noting that the total entropy of the isolated system must increase
\[ \Delta S = \frac{\Delta N}{T} (\mu_1 - \mu_2) > 0 \]
which implies that \( \mu_1 > \mu_2 \).

3) FALSE.

The correct statement is: at \( T=0 \), the total energy of a Bose gas is zero while the total energy of a Fermi gas is non-zero. This is a consequence of the Pauli exclusion principle which is obeyed by fermions but not obeyed by bosons.

For bosons at \( T=0 \), all particles can simultaneously occupy their single particle ground states. Defining the ground state energy to be zero, the total energy of the system is then zero.

For fermions at \( T=0 \), only \( 2s+1 \) particles can occupy their single particle ground states. Here, \( s \) is the spin of the fermion and \( 2s+1 \) is the degeneracy of each single-particle energy level. Thus, the total number of fermions in the gas \( N \geq 2s+1 \), and some must occupy excited energy levels.

An explicit calculation gives \( E = \frac{3}{5} N \varepsilon_F \) for the energy of an ideal Fermi gas at \( T=0 \), where the Fermi energy is
\[ \varepsilon_F = \frac{\hbar^2}{2m} \left( \frac{6\pi^2 N}{(2s+1)V} \right)^{1/3} \]

Clearly, \( E \neq 0 \).
4 TRUE.

Consider an isolated system at fixed temperature and pressure. The Gibbs free energy is defined as

\[ G = E + PV - TS \]

For an isolated system, the total energy \( E \) is fixed and the total volume is fixed. Thus, at constant \( P \) and \( T \),

\[ \Delta G = -T \Delta S \]

But, in the approach to equilibrium, \( \Delta S > 0 \) for an isolated system. Thus,

\[ \Delta G \leq 0 \]

That is, the Gibbs free energy is minimized in the approach to thermodynamic equilibrium at constant \( T \) and \( P \).

Note: The utility of this result is that it also applies to non-isolated systems as long as \( T \) and \( P \) of the system are held constant (e.g., by placing the system in thermal contact with a heat reservoir and movable piston).

5 TRUE.

For a classical ideal gas, the entropy is given by

\[ S = N \left[ \ln \left( \frac{V}{N}^{3/2} \right) + \frac{5}{2} \right] \]

where \( \lambda_{th} = \left( \frac{2 \pi m k}{kT} \right)^{1/2} \).
6) During the process where thermal equilibrium is reached, system 1 gains energy
\[ \Delta E_1 = \int_{T_1}^{T_f} C_v \, dT \]
while system 2 loses energy
\[ \Delta E_2 = \int_{T_2}^{T_f} C_v \, dT \]
where \( T_1 \) and \( T_2 \) are the initial temperatures of systems 1 and 2 respectively and \( T_f \) is the common final temperature.

Conservation of energy requires that
\[ \Delta E_1 + \Delta E_2 = 0 \]
Plugging in \( C_v = b \, T^3 \), we have
\[ \frac{b}{4} (T_f^4 - T_1^4) + \frac{b}{4} (T_f^4 - T_2^4) = 0 \]
Solving for \( T_f \),
\[ T_f = \left( \frac{T_1^4 + T_2^4}{2} \right)^{\frac{1}{4}} \]
Plugging in \( T_1 = 100 \, K \) and \( T_2 = 200 \, K \), one finds \( T_f = 170.7 \, K \).

Thus, the correct answer is (d).

7) The average speed of the molecules of an ideal gas is
\[ \langle v \rangle = \sqrt{\frac{8kT}{\pi m}} \]
Noting that \( m_B = 2m_A \),
\[ \frac{\langle v_B \rangle}{\langle v_A \rangle} = \frac{\sqrt{m_A}}{\sqrt{m_B}} = \sqrt{\frac{1}{2}} \]
Thus, the correct answer is (d).

8) In class, I showed that a necessary condition for mechanical stability is:
\[ \left( \frac{\partial V}{\partial P} \right)_T < 0 \]
For a van der Waals gas, the isotherms below the critical temperature look as follows:

Between the points A and B labeled above, \( \left( \frac{\partial V}{\partial P} \right)_T > 0 \). This, in this region, the van der Waals equation of state cannot represent a state of mechanical equilibrium.
This defect can be "repaired" by using the Maxwell construction. The resulting isotherm indicated by the solid line

\[ \text{(where the areas enclosed by region C and D are equal)} \]

provides a good model for the liquid-gas phase transformation.

All the other statements of the problem are correct. Thus, the correct answer is (c).

9. Given a solid and liquid co-existing in thermal equilibrium, the typical behavior is that the liquid freezes completely when the pressure increases. The water/ice phase transformations is one of the few counter-examples to this behavior.

The question posed is immediately answered by examining the Clausius-Clapeyron equation:

\[ \left( \frac{dP}{dT} \right)_{\text{melting curve}} = \frac{L_{\text{fusion}}}{T(V_{\text{liquid}} - V_{\text{solid}})} \]

If the liquid freezes when the pressure is raised, this means that \( \left( \frac{dP}{dT} \right)_{\text{melting curve}} > 0 \). Since the latent heat of fusion, \( L_{\text{fusion}} > 0 \) (it takes positive heat energy to melt the solid), it follows that \( V_{\text{liquid}} > V_{\text{solid}} \). Thus, during the freezing process, the specific volume decreases.

The correct answer is (b).

10. The Fermi temperature is related to the Fermi energy by \( E_F = kT_F \). Thus, for a spin-1/2 particle \( \left( s = \frac{1}{2} \right) \), we have \( 2s+1 = 2 \) and

\[ E_F = kT_F = \frac{\hbar^2}{2m} \left( \frac{3\pi^2 N}{V} \right)^{2/3} \]

First, we need to compute \( N/V \). For \(^3\text{He} \), \( g = 0.021 \text{ g cm}^{-3} \).

\[ \frac{g}{V} = \frac{N}{V} \frac{m_{\text{He}}}{m_{\text{He}}} = \frac{3}{4} \left( 6.699 \times 10^{-25} \text{ kg} \right) \]

\[ = 4.987 \times 10^{-22} \text{ kg} \]

Convert \( g \) to \( \text{kg m}^{-3} \):

\[ 1 \text{ g cm}^{-3} = 10^6 \text{ g m}^{-3} = 10^3 \text{ kg m}^{-3} \]

Thus,

\[ \frac{N}{V} = \frac{g}{m} = \frac{3}{4} \times \frac{31 \text{ kg m}^{-3}}{4.987 \times 10^{-22} \text{ kg}} = 1.624 \times 10^{28} \text{ m}^{-3} \]

Finally,

\[ T_F = \frac{(1.0546 \times 10^{-3} \text{ JK}^{-1})^2}{2 (1.381 \times 10^{-23} \text{ JK}^{-1})} \left( \frac{3\pi^2 (1.624 \times 10^{28} \text{ m}^{-3})}{} \right)^{2/3} \]

\[ = 4.96 \text{ K} \]

Thus, the correct answer is (d).
PART III: PROBLEMS

1. Use the Clausius-Clapeyron equation:

\[
\frac{dP}{dT} = \frac{L_{vap}}{T(T_{vap} - T_{eq})}
\]

[RB eq. (12.27) on p.281]. To a very good approximation, \(v_{vap} \ll v_{eq}\).

If we use the ideal gas law,

\[ P_{vap} = kT \]

(where \( v_{vap} = V/V_{N} \)), then

\[
\frac{dP}{dT} = \frac{P}{kT^2} L_{vap}
\]

Assume that \( L_{vap} \) is constant over the range \(98°C < T < 103°C\).
Then, this is an easy differential equation to solve. RB eq. (12.30) on p.281 provides the solution:

\[
P = \frac{P_{0} \exp \left( -\frac{L_{vap}}{kT} \frac{T}{T_{0}} \right)}{\exp \left( \frac{-L_{vap}}{kT_{0}} \right)}
\]

\[ = P_{0} \exp \left( \frac{-L_{vap}}{k} \left( \frac{T}{T_{0}} - \frac{1}{T_{0}} \right) \right) \]

Solving for \( L_{vap} \),

\[
L_{vap} = k \left( \frac{1}{T} - \frac{1}{T_{0}} \right)^{-1} \ln \left( \frac{P}{P_{0}} \right)
\]

We are given:

\[
T = 103°C \quad \text{at} \quad z = 0
\]

\[
T = 98°C \quad \text{at} \quad z = 0.5 \text{ km}
\]

where \( z \) is the altitude. Let us determine the corresponding pressures.
At \( z = 0 \), \( p = 1 \text{ atm} = 1.013 \times 10^{5} \text{ N/m}^2 \). At \( z = 0.5 \text{ km} \), if we employ the approximation of the isothermal atmosphere, then

\[
P(z) = P(0) e^{-mgz/kT}
\]

[See problem 4 on problem set #6: RB Chapter 7, problem 1.]

Taking the mass of an air molecule to be \[ \text{[RB, Table 5.2 on p.107]} \]

\[ m = 4.8 \times 10^{-26} \text{ kg} \]

and the atmospheric temperature to be \( T = 300 \),

\[
P(z = 0.5 \text{ km}) = (1 \text{ atm}) \exp \left( \frac{-(4.8 \times 10^{-26} \text{ kg}/(9.2 \text{ m/s}^2)/8.31 \text{ J K}^{-1})/300 \text{ K}}{1.38 \times 10^{-23} \text{ J K}^{-1}/300 \text{ K}} \right)
\]

\[ = 0.945 \text{ atm} \]

Plugging in:

\[
T = 103°C = 376 \text{ K} \quad \text{at} \quad P_0 = 1 \text{ atm}
\]

\[
T = 98°C = 371 \text{ K} \quad \text{at} \quad P_0 = 0.945 \text{ atm}
\]

into the equation for \( L_{vap} \):

\[
L_{vap} = \left( \frac{1.38 \times 10^{-23} \text{ J K}^{-1}}{376} - \frac{1}{371} \right)^{-1} \ln \left( \frac{0.945}{1} \right)
\]

\[ = 2.18 \times 10^{-20} \text{ J/molecule} \]
Note that $L_{\text{vap}}$ corresponds to the heat required to convert one mole of the liquid into gas. In terms of $eV$,

$$\frac{L_{\text{vap}}}{1.602 \times 10^{-19} \text{ J/eV}} = 2.18 \times 10^{-20} \text{ J}$$

L_{\text{vap}} = 0.136 \text{ eV/molecule}

This can be compared with, e.g., $L_{\text{vap}} = 0.42 \text{ eV/molecule}$ for water [see RB Table 12.2 on p 274].

(2) (a) At constant temperature and number of molecules, the multiplicity is proportional to $V^n$. [See RB section 2.4, pp 31-35].

The ratio of the final to the initial multiplicity is:

$$\frac{g_f}{g_i} = \left(\frac{V_f}{V_i}\right)^n = \left(\frac{0.1 \text{ m}^3}{0.3 \text{ m}^3}\right)^n = 3^{-n}$$

We can write:

$$3^{-n} = \left(10 \log_{10} 3\right)^{-n} = (10^{0.4771})^{-n}$$

For $N = 3 \times 10^{24}$ molecules,

$$\frac{g_f}{g_i} = 10^{-1.43 \times 10^{24}}$$

(b) The change in entropy of the gas is

$$\Delta S = k \ln g_f - k \ln g_i$$

$$= k \ln \left(\frac{g_f}{g_i}\right)$$

$$= k \ln 3^{-n}$$

$$= -nk \ln 3$$

$$= -(1.381 \times 10^{-23} \text{ JK}^{-1})(3 \times 10^{24}) \ln 3$$

$$= -45.5 \text{ J/K}$$
(c) By assumption, the compression takes place slowly, so
\[ \Delta S_{\text{gas}} + \Delta S_{\text{environment}} = 0 \]

Thus, from part (b),
\[ \Delta S_{\text{environment}} = -\Delta S_{\text{gas}} \]
\[ = 45.5 \text{ J/K} \]

In addition, at constant temperature \( T = 280 \text{ K} \), the heat energy added to the environment is
\[ q = T \Delta S_{\text{environment}} \]
\[ = (280 \text{ K})(45.5 \text{ J/K}) \]
\[ = 1.27 \times 10^4 \text{ J} \]

(d) There are two methods to compute the work done on the gas during compression.

**Method 1:**

The work done by the gas is equal to
\[ w = \int_{V_i}^{V_f} P \, dV \]

So the work done on the gas is equal to \(-w\). If we use the ideal gas law: \( P = \frac{NkT}{V} \), then since \( T \) is constant,
\[ -w = -\int_{V_i}^{V_f} \frac{NkT}{V} \, dV = -NkT \ln\left(\frac{V_f}{V_i}\right) \]

Using \( \frac{V_f}{V_i} = \frac{1}{3} \), we obtain
\[ -w = NkT \ln 3 \]
\[ = (3 \times 10^{29})(1.38 \times 10^{-23} \text{ JK}^{-1})(280 \text{ K}) \ln 3 \]
\[ = 1.27 \times 10^4 \text{ J} \]

**Method 2:**

For an ideal gas, the energy remains constant if the temperature is fixed. Using the first law of thermodynamics,
\[ q = \Delta E + w \]

where \( q \) is the heat energy added to the system (i.e., the gas) and \( w \) is the work done by the gas. Since \( \Delta E = 0 \),
\[ w = -q_{\text{gas}} \]

But \( q_{\text{gas}} \) is equal to the negative of the heat added to the environment as computed in part (b), since this is heat removed from the gas. Hence,
\[ q_{\text{gas}} = -1.27 \times 10^4 \text{ J} \]

and indeed we confirm that the work done on the gas is
\[ w = 1.27 \times 10^4 \text{ J}. \]
3. (a) The "statistical" definition of temperature is:

\[ \frac{1}{T} = \left( \frac{\partial S}{\partial E} \right)_{\text{fixed external parameters}} \]

[see RB eq. (4.21) on p. 82].

We are given for the black hole

\[ S = \frac{\mathcal{A} c^3 A}{4 G M} \]

where \( A = 4 \pi R_s^2 \) and \( R_s = \frac{2GM}{c^2} \). Thus,

\[ S = \frac{4 \pi \mathcal{A} G M^2}{c} \]

Now, \( M \) is related to the total internal energy of the black hole via \( E = MC^2 \). Thus,

\[ S = \frac{4 \pi \mathcal{A} C^2 E^2}{c^5} \]

We can now apply the statistical definition of temperature:

\[ \frac{1}{T} = \frac{8 \pi \mathcal{A} C E}{c^5} \]

Expressing this result in terms of the black hole mass (using \( E = MC^2 \) again),

\[ T = \frac{\mathcal{A} c^3}{8 \pi \mathcal{A} G M} \]

(b) Let us return to the expression for \( E \) in terms of the temperature \( T \). We can write this as:

\[ E = \frac{\mathcal{A} c^3}{8 \pi \mathcal{A} G M T} \]

The heat capacity is

\[ C = \frac{\partial E}{\partial T} = -\frac{\mathcal{A} c^3}{8 \pi \mathcal{A} G M T^2} \]

Thus, the heat capacity is negative! This means that as the black hole loses energy, its temperature increases! How bizarre!

(c) The Stefan-Boltzmann law states that the energy flux of a blackbody radiator is given by \( \sigma B T^4 \). [see RB eq. (6.30) on p. 124]

Energy flux = \( \frac{dE}{dh \Delta t} \)

If we integrate over the surface area of the black hole, we obtain the power radiated at the surface is equal to

\[ \frac{dE}{dt} = -\sigma A_0 B T^4 \]

(The minus sign indicates that energy is being lost in radiation)

Plug in \( E = MC^2 \) on the left hand side and

\[ A = 4 \pi R_s^2 = 4 \pi \left( \frac{2GM}{c^2} \right)^2 = \frac{16 \pi G^2 M^2}{c^4} \]

\[ T = \frac{\mathcal{A} c^3}{8 \pi \mathcal{A} G M} \quad \text{[obtained in part (a)]} \]

On the right hand side
\[ c^2 \frac{dM}{dt} = -\frac{16\pi G^2 M^2}{c^4} \sigma_B \left( \frac{\hbar c^3}{8\pi \hbar \kappa M} \right)^4 \]

Simplifying,
\[ \frac{dM}{dt} = -\frac{\sigma_B \kappa^4 c^6}{256 \pi^3 \hbar^4 \kappa^2} \frac{1}{M^2} \]

We can simplify this expression by noting that [RS eq (6.29) on p124] that the Stefan-Boltzmann constant can be expressed in terms of fundamental constants:
\[ \sigma_B = \frac{\pi^2 k^4}{60 \kappa^4 c^2} \]

Thus,
\[ \frac{dM}{dt} = -\frac{\kappa c^4}{15 \cdot 2^{10} \pi G^2 M^2} \]

(a) Solving the differential equation,
\[ \int M^2 \frac{dM}{dt} = -\frac{\kappa c^4}{15 \cdot 2^{10} \pi G^2} \int dt \]
\[ \frac{1}{3} (M_f^3 - M_i^3) = -\frac{\kappa c^4}{15 \cdot 2^{10} \pi G^2} (t_f - t_i) \]

where \( i \) and \( f \) refer to the initial and final states.

If the black hole completely evaporates, then \( M_f = 0 \).
This will occur in a time interval
\[ \Delta t = \frac{5.2 \times 10^9 \pi G^2 M^3}{\kappa c^4} \]

where \( M \) is the mass of the initial black hole.

If \( M = 2 \times 10^{11} \text{ kg} \),
\[ \Delta t = \frac{5.2 \times 10^9 \pi (6.673 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2})^2 (2 \times 10^{11} \text{ kg})^3}{(1.0546 \times 10^{-34} \text{ J} \cdot \text{s}) (3 \times 10^8 \text{ m} \cdot \text{s}^{-1})^2} \]

or
\[ \Delta t = 6.71 \times 10^{13} \text{ sec}. \]

Remark: In black hole physics, it is often useful to express the fundamental constants in terms of the Planck mass, which is defined as
\[ M_P = \sqrt{\frac{\hbar c}{G}} = 2.18 \times 10^{-8} \text{ kg} \]

Then, it is convenient to replace Newton's constant \( G \) in favor of \( M_P \). For example, the temperature of a black hole is
\[ T = \frac{\kappa c}{8\pi \hbar \kappa M} M_P^2 = \frac{M_P^2 c^2}{8\pi \hbar \kappa M} \]

Note that if \( M = M_P \), then \( \Delta T = \frac{M_P c^2}{8\pi \hbar \kappa M} \). That is, for a black hole the total energy is of \( O(\Delta T) \) only when \( M \sim 0(M_P) \).