1. Show that for complex scalar fields,

\[
\int \mathcal{D}\Phi^* \mathcal{D}\Phi \exp \left\{ i \int d^4x d^4y \left[ \Phi^*(x)M(x,y)\Phi(y) \right] + i \int d^4x \left[ J^*(x)\Phi(x) + \Phi^*(x)J(x) \right] \right\} = \mathcal{N} \frac{1}{\det M} \exp \left\{ -i \int d^4x d^4y J^*(x)M^{-1}(x,y)J(y) \right\},
\]

for some infinite constant \( \mathcal{N} \). This is problem 14.1 on p. 283 of Schwartz.

2. (a) Derive the result:

\[
\int d^4z \frac{\delta^2 W[J]}{\delta J(x)\delta J(z)} \frac{\delta^2 \Gamma[\Phi]}{\delta \Phi(z)\delta \Phi(y)} = -\delta^4(x - y),
\]

and interpret diagrammatically. Here, \( W[J] \) is the generating functional for the connected Green functions and \( \Gamma[\Phi] \) is the generating functional for the one particle irreducible (1PI) Green functions.

(b) By taking one further functional derivative, show that \( \Gamma \) generates the amputated connected three-point function.

3. Consider the quantum field theory of a real scalar field governed by the Lagrangian,

\[
\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4.
\]

(a) Evaluate the generating functional \( Z[J] \) perturbatively, keeping all terms up to and including terms of \( \mathcal{O}(\lambda) \) as follows. First, show that \( Z[J] \) can be written in the following form,

\[
Z[J] = \mathcal{N} \left[ 1 - \frac{i\lambda}{4!} \int d^4y \left( \frac{1}{i \delta J(y)} \right)^4 + \mathcal{O}(\lambda^2) \right] \exp \left\{ -\frac{i}{2} \int d^4x_1 d^4x_2 J(x_1) \Delta_F(x_1 - x_2) J(x_2) \right\},
\]

where \( \mathcal{N} \) is the \( J \)-independent constant. Then, carry out the functional derivatives with respect to \( J \), keeping all terms up to and including terms of \( \mathcal{O}(\lambda) \). Using the result just obtained for \( Z[J] \), obtain an expression for the generating functional for the connected Green functions, \( W[J] \), keeping all terms up to and including terms of \( \mathcal{O}(\lambda) \).
(b) Using the result of part (a) for $W[J]$, compute the four-point connected Green function. Check that the same result is obtained by making use of Coleman’s lemma derived in class to obtain the $O(\lambda)$ contribution to $G^{(4)}(x_1, x_2, x_3, x_4)$. By taking the appropriate Fourier transform, verify that you obtain the momentum space Feynman rule for the four-point scalar interaction obtained in class.

(c) Evaluate the classical field $\phi_c(x)$ and the generating functional for the 1PI Green functions, $\Gamma[\phi_c]$, perturbatively, keeping all terms up to and including terms of $O(\lambda)$. Then, repeat part (b) for the four-point 1PI Green function.

4. Consider a scalar field theory defined by the Lagrangian density

$$\mathcal{L} = \frac{1}{2} \partial^\mu \phi(x) \partial_\mu \phi(x) - V(\phi(x)), \quad (2)$$

and the corresponding equation of motion,

$$\square \phi(x) + V'(\phi) = 0,$$

where $\square \equiv \partial^\mu \partial_\mu$ and $V' \equiv dV/d\phi$.

(a) Starting from eq. (14.122) on p. 276 of Schwartz, derive the equation of motion for the Green function $\langle \Omega | T \{ \phi(x) \phi(y) \} | \Omega \rangle$,

$$\square \langle \Omega | T \{ \phi(x) \phi(y) \} | \Omega \rangle = -\langle \Omega | T \{ V'(\phi(x)) \phi(y) \} | \Omega \rangle - i \delta^4(x - y). \quad (3)$$

(b) Derive eq. (3) by the following technique. Start from the path integral definition of the generating functional,

$$Z[J] = N \int \mathcal{D}\phi \exp \left\{ i \int d^4x \left[ \mathcal{L} + J(x)\phi(x) \right] \right\}, \quad (4)$$

where $N$ is chosen such that $Z[0] = 1$. Perform a change of variables in the path integral, $\phi(x) \to \phi(x) + \varepsilon(x)$, where $\varepsilon(x)$ is an arbitrary infinitesimal function of $x$. Noting that a change of variables does not change the value of of $Z[J]$, show that to first order in $\varepsilon(x)$,

$$\int \mathcal{D}\phi \exp \left\{ i \int d^4x \left[ \mathcal{L} + J(x)\phi(x) \right] \right\} \int d^4x \varepsilon(x) \left[ -\square \Phi - V'(\phi) + J(x) \right] = 0. \quad (5)$$

Since $\varepsilon(x)$ is arbitrary, we may choose $\varepsilon(x) = \epsilon \delta^4(x - y)$, where $\epsilon$ is an infinitesimal constant. With this choice for $\varepsilon(x)$, show that by taking the functional derivative of the eq. (5) with respect to $J(x)$ and then setting $J = 0$, one can derive eq. (3).

HINT: What is the Jacobian corresponding to the change of variables, $\phi(x) \to \phi(x) + \varepsilon(x)$?

\footnote{Just as in the case of ordinary functional integration, a change of integration variables does not change the value of the functional integral.}