Physics 101A  Modern Physics  Winter 1999

Homework Assignment #7

Due in class, Thursday, March 4.  90 points total.
I will accept the homework late up until noon Friday, March 5, in my office.

1) (10 pnts) Suppose that you superpose a continuous set of cosine waves, \( \cos kx \), with this distribution of amplitudes:

\[
A(k) = \begin{cases} 
0 & k < k_0 - a \\
1/(2a) & k_0 - a < k < k_0 + a \\
0 & k > k_0 + a 
\end{cases}
\]

a) Graph this function \( A(k) \).

b) Show that the explicit spatial form of the 1-dimensional wave packet obtained by adding the cosine waves for all possible values of \( k \) according to the integral

\[
\psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) \cos(kx) \, dk
\]

is

\[
\psi(x) = \frac{1}{\sqrt{2\pi}} \cos k_0 x \sin ax \frac{1}{ax}
\]

(You can think of this integral as a sum of an (uncountably) infinite number of cosine terms, \( \cos kx \), each of which is multiplied by an amplitude factor \( A(k)dk \) before summing.  Technically it is a Fourier transform, but you don’t need to know anything about integral transforms to be able to do the integral and have some idea of what it means.)

c) Graph this wave packet for \( k_0 = 0 \), noting on your graph the values of \( x \) closest to the origin where the wave packet amplitude is zero.

d) Show roughly that the width of the packet \( \psi(x) \) and the width of \( A(k) \) satisfy the relation \( \Delta x \Delta k > 1 \), independent of the value of \( a \). This need not be rigorous—just make reasonable estimates of \( \Delta x \) and \( \Delta k \) from your graphs.

2) (10 pnts) For electromagnetic waves in a waveguide (a conducting pipe) of rectangular cross section, the wave number and the frequency are related by the dispersion relation

\[
k = \sqrt{\frac{\omega^2}{c^2} - \left( \frac{m\pi}{a} \right)^2 - \left( \frac{n\pi}{b} \right)^2}
\]

where \( a \) and \( b \) are the rectangular dimensions of the pipe, and \( m \) and \( n \) are integers that characterize the mode of oscillation of the wave in the pipe.

a) What is the phase velocity of the waves?

b) What is the group velocity?

c) Show that \( v_p > c \) and \( v_g < c \).

d) Show that \( v_g v_p = c^2 \).
3) (10 pnts) For a relativistic particle, the equations for the frequency and the de Broglie wavelength become

\[ \nu = \frac{E}{h} = \frac{mc^2/h}{\sqrt{1 - v^2/c^2}} \]
\[ \lambda = \frac{h}{p} = \frac{h \cdot \sqrt{1 - v^2/c^2}}{mv} \]

(The rest energy is included in the energy \( E \). Because of this, the frequency for a particle at rest is not zero but is \( mc^2/h \). This quantity plays the role of an additive constant in the frequency; it has no observable consequences.) Show that the phase velocity and the group velocity of the de Broglie waves for such a particle are

\[ v_p = \frac{c^2}{\nu} \quad \text{and} \quad v_g = \nu. \]

4) (10 pnts) Question 3-16 in Eisberg & Resnick.

5) (10 pnts) Question 5-8 in Eisberg & Resnick and Problem 5-1.

6) (10 pnts) As demonstrated in the previous problem, a valid solution to the Schrödinger equation can be obtained by adding together two or more solutions. Suppose that \( \psi_1(x,t) \) and \( \psi_2(x,t) \) are two possible normalized and orthogonal wave functions, with corresponding energies \( E_1 \) and \( E_2 \), for a particle in a certain potential (thus they are “energy eigenstates”). That is,

\[ \int_{-\infty}^{+\infty} \psi_i^*(x,t) \psi_j(x,t) dx = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \]

Now suppose that the particle happens to be in a state that is a superposition \( \psi(x,t) = a \psi_1(x,t) + b \psi_2(x,t) \), where \( a \) and \( b \) are complex constants, such that

\[ |a|^2 + |b|^2 = 1. \]

a) Show that the wave function \( \psi(x,t) \) is properly normalized. That is,

\[ \int_{-\infty}^{+\infty} \psi^*(x,t) \psi(x,t) dx = 1. \]

b) If the energy of the particle is measured, the probability that the value \( E_1 \) is obtained is given by the absolute value squared of

\[ \int_{-\infty}^{+\infty} \psi_1^*(x,t) \psi(x,t) dx. \]

Show this is equal to \( |a|^2 \), while the probability of measuring \( E_2 \) is equal to \( |b|^2 \).

c) Derive an expression for the average energy in terms of \( E_1, E_2, a \) and \( b \).

7) (10 pnts) Problem 5-2 in Eisberg & Resnick.

8) (20 pnts) Problems 9, 10, 11, 12, 13 in Chapter 5 of Eisberg and Resnick. These problems together are the same as the calculation in Example 5-9, only with a slightly different wave function.

Required Reading: Eisberg & Resnick, Chapter 5.

Recommended Reading: Chapters 40-41 of Young & Freedman, “University Physics.”