Name: Solutions

Physics 101B Modern Physics Spring 1998

Midterm Exam
April 30, 1999

Some physical constants:
- Speed of light: $c = 3.00 \times 10^8$ m/s.
- Stefan-Boltzmann constant: $\sigma = 5.67 \times 10^{-8}$ W/(m$^2$K$^4$).
- Boltzmann’s constant: $k = 8.62 \times 10^{-5}$ eV/K.
- Planck’s constant: $\hbar = 6.63 \times 10^{-34}$ Js, $\hbar c = 1970$ eVÅ

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1) (4 pts) A single beam of neutral sodium atoms (Z=11) in the ground state passes through a Stern-Gerlach magnet. How many beams emerge from the magnet?
   a) 1  
   b) 2  
   c) 3  
   d) 4  
   \[ j = \frac{1}{2} \]

2) (4 pts) The ground state (1S) of the hydrogen atom is actually split into two closely spaced levels (the “hyperfine” structure). A transition from the upper level to the lower level results in the emission of a radio-band photon with a wavelength of 21 cm. The splitting is due to
   a) Quantum fluctuations in the electromagnetic field (the “Lamb shift”).
   b) The interaction of the electron spin magnetic moment with the magnetic field produced by its orbital motion.
   c) The interaction between the electron spin magnetic moment and the spin magnetic moment of the proton.
   d) The Zeeman effect.

3) (4 pts) The solution of the Dirac equation for an electron in a coulomb potential predicts for the fine structure of the $n = 2, j = \frac{1}{2}$ level that the $\ell = 0$ and $\ell = 1$ states have exactly the same energy. In fact, however, the $\ell = 0$ state has $4.4 \times 10^{-6}$ eV higher energy than the $\ell = 1$ state. This splitting is due to
   a) Quantum fluctuations in the electromagnetic field (the “Lamb shift”).
   b) The interaction of the electron spin magnetic moment with the magnetic field produced by its orbital motion.
   c) The interaction between the electron spin magnetic moment and the spin magnetic moment of the proton.
   d) The Zeeman effect.

4) (4 pts) Which of the following coulomb-potential wave functions has a non-zero uncertainty in the $z$ component of the orbital angular momentum, $L_z$? The notation is $\psi_{n,\ell, m_{\ell}}$.
   a) $\psi(\vec{r}) = \psi_{3,2,-1}(r, \theta, \phi)$
   b) $\psi(\vec{r}) = \psi_{3,2,0}(r, \theta, \phi)$
   c) $\psi(\vec{r}) = \frac{1}{\sqrt{5}} \cdot \psi_{2,1,1}(r, \theta, \phi) + \frac{2}{\sqrt{5}} \cdot \psi_{3,1,1}(r, \theta, \phi)$
   d) $\psi(\vec{r}) = \frac{2}{\sqrt{13}} \cdot \psi_{3,2,1}(r, \theta, \phi) + \frac{3}{\sqrt{13}} \cdot \psi_{3,2,2}(r, \theta, \phi)$

$m_{\ell}$ is $n + \ell$, $n \geq \ell$.
5) (10 pts) What are the expectation values of $L^2$ and $L_z$ for the normalized coulomb-potential wave function $\psi(\vec{r}) = \frac{3}{5} \psi_{2,1,1}(r, \theta, \phi) + \frac{4}{5} \psi_{2,1,0}(r, \theta, \phi)$?

$$\langle L^2 \rangle = \ell (\ell + 1) \hbar^2 = 1 (1+1) \hbar^2 = 2 \hbar^2$$

$$\langle L_z \rangle = \left( \frac{3}{5} \right)^2 m_g \hbar + \left( \frac{4}{5} \right)^2 m_g \hbar$$

$$= \frac{9}{25} m_g \hbar + \frac{16}{25} 0 \hbar$$

$$= \frac{9}{25} \hbar$$

6) (10 pts) Suppose that $\ell = 2$ is the eigenvalue of the total orbital angular momentum of an atom and $s = \frac{1}{2}$ is the eigenvalue of the total spin angular momentum.

a) List all of the possible values of $j$, the eigenvalue of the total angular momentum $\vec{J} = \vec{L} + \vec{S}$.

$$j = |\ell - s| + s \ell + s$$

$$= \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}$$

b) Suppose that it is known that the eigenvalues of $L_z$ and $S_z$ are respectively $m_L = 2$ and $m_s = -\frac{1}{2}$. List all of the possible values of $j$ and of $m_j$, the eigenvalue of $J_z$.

$$m_j = m_L + m_s - \ell - \frac{1}{2} = \frac{3}{2}$$

$$|m_j| \leq j$$

so

$$j = \frac{1}{2}, \frac{3}{2}, \text{ or } \frac{7}{2}$$
7) (10 pts) The figure on the left below shows the fine structure of the first excited state of the lithium atom \((Z=3)\). Label the two energy levels according to their spectroscopic notation (which specifies the quantum numbers for spin, orbital angular momentum, and total angular momentum). The figure on the right displays the same levels when atom is placed in an external magnetic field. What quantum number distinguishes the levels in the two multiplets? Label each of the levels according to that quantum number.

\[
\begin{align*}
\text{Ground state} & : 1s^2 2s \\ 
\text{1st excited} & : 1s^2 2\rho
\end{align*}
\]

Without magnetic field

\[
\begin{align*}
\frac{\hbar}{2} & \frac{3\hbar}{2} \\
\frac{1\hbar}{2} & \frac{-1\hbar}{2}
\end{align*}
\]

With magnetic field

\[
\begin{align*}
m_j = \frac{3\hbar}{2} & \\
m_j = \frac{\hbar}{2} & \\
m_j = -\frac{\hbar}{2} & \\
m_j = -\frac{3\hbar}{2}
\end{align*}
\]

8) (9 pts) Explain why for multi-electron atoms it makes sense to use the same quantum numbers \(\ell, m_\ell\) and, as a good first approximation, the same angular functions \(Y_{\ell m}(\theta, \phi)\) as were derived in the solution of the hydrogen atom.

These angular functions and quantum numbers apply to any central potential \(V(r)\) (i.e. spherically symmetric). Since all filled shells are spherically symmetric, only the outermost, most energetic few electrons in any atom can violate this symmetry. Therefore, it is a good approximation to assume spherically symmetry when solving the S.E. for a multi-electron atom.
9) (10 pts) The figure below shows the $n = 2$ and $n = 3$ energy levels of the optically active electron of the lithium atom. The fine structure is not shown.
   a) Indicate by lines between pairs of energy levels all of the allowed electric-dipole transitions.
   b) Circle all of the levels that are, in reality, fine-structure doublets.

For the doublets,
\[ \Delta = \ell - \frac{\ell}{2} \text{ or } \ell + \frac{\ell}{2} \]

10) (10 pts) The titanium atom (Ti) has $Z = 22$ and is one of the transition elements.
   a) Write down the electronic configuration for Ti (for example, boron, with $Z = 5$, is $(1s)^2(2s)^22p$). There are no exceptions to the standard ordering of energy levels for this element.

\[
\begin{align*}
|s^2 2s^2 2\rho^6 2s^2 3\rho^6 4s^2 3d^2
\end{align*}
\]

b) What are the values of the quantum numbers $L$, $S$, and $J$ in the ground state of titanium? Note that there are no exceptions to Hund's rules ($LS$ coupling rules) for this element.

\[
2 \text{ electrons with } s = \frac{1}{2}, l = 2
\]

\[M_{l_{\text{max}}} = 1 \Rightarrow s' = 1\]

\[M_{S_{\text{max}}} = l + s = 3 \Rightarrow \ell' = 3 \]

\[J_{\text{min}} = 3 - 1 = 2\]

\[s' = 1\]

\[\ell = 3\]

\[j' = 2\]

\[J_{\text{total}} = \frac{2s + 1}{2} \]

\[ = 3\]

\[F_{2}\]

\[2s + 1 \frac{\ell}{J} = 3 F_{2}\]
11) (25 pts) Two electrons are in a cubical box (a 3-D infinite square well potential) with sides of length \(a\). As you know from a homework assignment, the spatial eigenfunctions for a single electron in such a box are

\[
\psi_{\ell mn}(\vec{r}) = \left(\frac{2}{a}\right)^{3/2} \sin \frac{\ell \pi x}{a} \sin \frac{m \pi y}{a} \sin \frac{n \pi z}{a},
\]

with energy \(E_{\ell mn} = \frac{\hbar^2 \pi^2}{2m_e a^2} (\ell^2 + m^2 + n^2)\). Use the notation \(|\uparrow\rangle_i\) for the spin eigenfunction of the \(i\)th electron with \(s = \frac{1}{2}\), \(m_s = +\frac{1}{2}\) and \(|\downarrow\rangle_i\) for the spin eigenfunction with \(s = \frac{1}{2}\), \(m_s = -\frac{1}{2}\).

a) Assuming that the interaction of the electrons with each other is negligible, write down a suitable wave function for the ground state of this system. Use the notation \(\psi_{\ell mn}(\vec{r})\), with appropriate values for \(\ell mn\), rather than writing out all of the \(\sin \frac{\ell \pi x}{a}\) functions.

\[
\Psi_0 = \psi_{111}(\vec{r}_1) \psi_{111}(\vec{r}_2) \frac{1}{\sqrt{2}} \left[ |\uparrow\rangle_1 |\downarrow\rangle_2 - |\downarrow\rangle_1 |\uparrow\rangle_2 \right]
\]

\[\text{symmetric} \quad \text{antisymmetric} \quad s = 0, m' = 0\]

b) Write down and label two wave functions for the first excited state of this system, one with \(s' = 0\), \(m' = 0\) and the other with \(s' = 1\), \(m' = 0\).

\[
\Psi = \frac{1}{\sqrt{2}} \left[ \psi_{112}(\vec{r}_1) \psi_{111}(\vec{r}_2) + \psi_{111}(\vec{r}_1) \psi_{112}(\vec{r}_2) \right] \frac{1}{\sqrt{2}} \left[ |\uparrow\rangle_1 |\downarrow\rangle_2 - |\downarrow\rangle_1 |\uparrow\rangle_2 \right]
\]

\[s = 0, m' = 0\]

\[
\Psi = \frac{1}{\sqrt{2}} \left[ \psi_{112}(\vec{r}_1) \psi_{111}(\vec{r}_2) - \psi_{111}(\vec{r}_1) \psi_{112}(\vec{r}_2) \right] \frac{1}{\sqrt{2}} \left[ |\uparrow\rangle_1 |\uparrow\rangle_2 + |\downarrow\rangle_1 |\downarrow\rangle_2 \right]
\]

\[s = 1, m' = 0\]

Continued on next page...
c) If the coulomb repulsion between the electrons were taken into account, which of the excited states would have the lower energy, the one with \( s' = 0 \), or the one with \( s' = 1 \)? Explain why.

\[ s' = 1 \text{ would have the lower energy because the spatial wave function is antisymmetric.} \]

The antisymmetric function keeps the electrons further apart on average than in the symmetric function, thereby resulting in a lower energy from coulomb repulsion.

d) If 7 electrons were put into the box, what would be the energy of the ground state in terms of \( h \), \( m_e \), and \( a \) (still assuming that the coulomb repulsion is negligible)?

2 electrons in ground state \( l = n = n = 1 \)
5 electrons in 1st excited state

\[ E = \frac{\pi^2 h^2}{2m_e^2} \left[ 2 \times [1+1+1] + 5 \times [1+1+2] \right] \]

\[ = 2 \times 3 + 5 \times 6 \]

2 of \( lmn = 1 \) \( \implies 3 \) possibilities

and 1 of \( lmn = 2 \) \( \times 2 \) spin states

\[ E = \frac{18\pi^2 h^2}{m_e^2} \]

Energy levels:
- \( 1\uparrow \uparrow \uparrow 1 \uparrow \uparrow \uparrow \uparrow \): 122, 212, 121
- \( 1\uparrow \downarrow 1 \uparrow \downarrow \downarrow \downarrow \): 112, 121, 211
- \( 1 \uparrow \downarrow \): 111