Name: Solutions

Physics 101B

Modern Physics

Spring 1998

Midterm Exam

May 7, 1998

Some physical constants:
- Speed of light: \( c = 3.00 \times 10^8 \text{ m/s} \).
- Stefan-Boltzmann constant: \( \sigma = 5.67 \times 10^{-8} \text{ W/(m}^2\text{K}^4) \).
- Boltzmann’s constant: \( k = 8.62 \times 10^{-5} \text{ eV/K} \).
- Planck’s constant: \( h = 6.63 \times 10^{-34} \text{ J s}, \ h\sigma = 1970 \text{ eV } \text{Å} \).
- Energies of the hydrogen atom levels: \( E = -13.6 \text{ eV} / n^2 \).

1) (3 pnts) Which one of the following wave functions has a nonzero expectation value for the electric dipole moment? The \( \psi_{n,l,m} \) are the eigenstates of a Coulomb potential. \( \Delta \ell = \pm 1 \)

a) \( \psi(\vec{r}) = \psi_{2,1,0}(r, \theta, \phi) \) \( \quad \psi(\vec{r}) = \frac{1}{\sqrt{5}} \cdot \psi_{2,1,0}(r, \theta, \phi) + \frac{2}{\sqrt{5}} \cdot \psi_{1,0,0}(r, \theta, \phi) \)

b) \( \psi(\vec{r}) = \psi_{2,1,-1}(r, \theta, \phi) \) \( \quad \psi(\vec{r}) = \frac{1}{\sqrt{2}} \cdot \psi_{3,2,0}(r, \theta, \phi) + \frac{1}{\sqrt{2}} \cdot \psi_{2,0,0}(r, \theta, \phi) \)

d) \( \psi(\vec{r}) = \psi_{3,2,0}(r, \theta, \phi) + \frac{1}{\sqrt{2}} \cdot \psi_{2,0,0}(r, \theta, \phi) \)

2) (3 pnts) Which of the following wave functions has a non-infinite uncertainty in the azimuthal angle (i.e. \( \Delta \phi \neq \infty \))?

a) \( \psi(\vec{r}) = \psi_{2,1,-1}(r, \theta, \phi) \) \( \psi(\vec{r}) = \frac{1}{\sqrt{5}} \cdot \psi_{2,1,1}(r, \theta, \phi) + \frac{2}{\sqrt{5}} \cdot \psi_{3,1,1}(r, \theta, \phi) \)

b) \( \psi(\vec{r}) = \psi_{2,1,1}(r, \theta, \phi) \) \( \psi(\vec{r}) = \frac{1}{\sqrt{5}} \cdot \psi_{2,1,1}(r, \theta, \phi) + \frac{2}{\sqrt{5}} \cdot \psi_{2,1,-1}(r, \theta, \phi) \)

d) \( \psi(\vec{r}) = \psi_{3,1,1}(r, \theta, \phi) + \frac{1}{\sqrt{2}} \cdot \psi_{2,0,0}(r, \theta, \phi) \)

For \( \Delta \phi \neq \infty \) we need \( \Delta L_z \neq 0 \) i.e. Not eigenstates of \( L_z \)

3) (5 pnts) What is the expectation value of \( L_z^2 \) for the normalized wave function

\( \psi(\vec{r}) = \frac{1}{\sqrt{5}} \cdot \psi_{2,1,1}(r, \theta, \phi) + \frac{2}{\sqrt{5}} \cdot \psi_{2,0,0}(r, \theta, \phi) \)

\( \langle L_z^2 \rangle = \left( \frac{1}{\sqrt{5}} \right)^2 (1)(l+1) + \left( \frac{2}{\sqrt{5}} \right)^2 (0)(0+1) \)

\( = \frac{2}{5} \hbar^2 \)

Since \( L_z \psi_{nlm} = l(l+1) \frac{\hbar^2}{\hbar} \psi_{nlm} \)
4) (8 pnts) Suppose that $L = 3$ is the eigenvalue of the total orbital angular momentum of an atom and $S = 2$ is the eigenvalue of the total spin angular momentum.
   a) List all of the possible values of $J$, the eigenvalue of the total angular momentum.
      \[ J = |L - S|, |L - S| + 1, \ldots, L + S \]
      \[ J = 1, 2, 3, 4, 5 \]
   b) Suppose that it is known that the eigenvalues of $L_z$ and $S_z$ are respectively $M_L = 2$ and $M_S = 1$. List all of the possible values of $J$ and of $M_J$, the eigenvalue of $J_z$.
      \[ M_J = M_L + M_S = 3 \]
      \[ J \geq |M_J| \quad \text{so} \quad J = 3, 4, 5 \]

5) (6 pnts) A beam of neutral sodium atoms is passed through a Stern-Gerlach type magnet with $B$ in the $+\hat{z}$ direction in the symmetry plane of the magnet and $\partial B_z/\partial z > 0$.
   a) The beam is split into $n_1$ parts along the $z$ direction. What is the value of $n_1$?
      For sodium, $J = \frac{1}{2}$ so $n_1 = 2J + 1 = 2$ parts
   b) Suppose that all but the uppermost of the emerging beams are blocked, while the uppermost beam is allowed to pass through a second magnet, which is identical in construction and orientation to the first. How many beams emerge from it?
      The upper beam has only one value of $M_S$, so it is not further split by the second magnet. $\Rightarrow$ 1 beam
   c) Suppose that neutral magnesium atoms are used instead of sodium. In that case how many beams emerge from the first magnet?
      For magnesium, $J = 0$ (closed 5 shell) so there is no splitting $\Rightarrow$ 1 beam
6) (10 pts) In the lowest-energy excited states of a sodium atom the valence electron is in a 3p orbital. These states are grouped into \( n \) energy levels of only slightly different energy ("fine structure").
   a) What is the value of \( n \)? \( L=1 \) and \( S = \frac{1}{2} \)
   So \( J = \frac{1}{2} \) or \( \frac{3}{2} \) Spin-orbit coupling \( \Rightarrow 2 \) levels
   b) What is the value of the quantum number \( J \) of the states in the level that has the lowest energy?
      \( J = \frac{1}{2} \)
   c) Describe in a few complete sentences the interaction that causes this energy splitting.
      This is due to the spin-orbit coupling.
      The magnetic moment of the electron interacts with the magnetic field produced by the orbital motion, resulting in an energy that depends on the sign of \( \vec{\mu} \cdot \vec{B} \) or, equivalently, \( \vec{S} \cdot \vec{L} \)

7) (4 pts) List two properties of liquid helium that change drastically when the temperature is lowered below the \( \lambda \) point, and give the direction in which the properties change.
   Viscosity decreases
   Thermal conductivity increases

8) (3 pts) During operation of a laser, the populations of the energy levels of the atoms responsible for emission of the laser light are in thermal equilibrium at a temperature \( T \) such that \( kT \) is much greater than the energy of a photon of the laser light.
   a) TRUE  Laser action cannot occur in thermal equilibrium
   b) FALSE

9) (3 pts) In a laser employing a four-level system, the stimulated emission takes place from a metastable excited state to a lower-lying excited state.
   a) TRUE
   b) FALSE
10) (9 pts) The germanium atom has Z = 32.
   a) Write down the electronic configuration (for example, boron, with Z = 5, is (1s)^2 (2s)^2 2p).
      \[ (1s)^2 (2s)^2 (2p)^6 (3s)^2 (3p)^6 (4s)^{\alpha} (3d)^{\beta} (4p)^\gamma \]  
      \[ \text{valence electrons} \]
      \[ \text{valence electrons} \]
   b) What are the values of the quantum numbers \( L, S, \) and \( J \) in the ground state of germanium?
      \[ M_s^{\text{max}} = \frac{1}{2} + \frac{1}{2} = 1 \quad \Rightarrow \quad S = 1 \]
      \[ m_l^{\text{max}} = 0 + 1 = 1 \quad \Rightarrow \quad L = 1 \]
      \[ \sum m_l = 0 \quad \Rightarrow \quad J = 0 \]
   c) Write down the spectroscopic notation for the ground state.
      \[ 2\ell + 1 \quad \ell = 3 \quad \ell \]

11) (10 pts) A gas of hydrogen atoms is in thermal equilibrium. What must be the temperature of the gas for there to be one half the number of atoms in the first excited state as in the ground state?

\[ \frac{e^{-E_2/hT}}{e^{-E_1/hT}} = \frac{1}{2} \]

\[ e^{(E_2 - E_1)/kT} = 2 \]

\[ (E_2 - E_1)/kT = \ln 2 \]

\[ T = \frac{1}{k} \frac{E_2 - E_1}{\ln 2} = \frac{1}{8.62 \times 10^{-5} \text{eV/K}} \frac{10.2 \text{ eV}}{\ln 2} \]

\[ T = 170,000 \degree K \]
12) (16 pts) Consider an ideal gas of $N$ non-interacting spin-$1/2$ fermions of mass $m$ in a one-dimensional infinite square well of width $L$. The following is a 1-D (i.e. simple) version of a calculation done in 3-D in the text and lecture.

a) Derive an expression for the set of all allowed energies of a single such particle
(this is a review of Physics 101A).

$$\psi = A \sin \frac{nkx}{L} \quad \psi(L) = 0 \Rightarrow kL = n\pi$$

$$k_n = \frac{n\pi}{L}$$

$$E = \frac{\hbar^2 k_n^2}{2m} = \frac{\hbar^2 \pi^2}{2mL^2} n^2 \quad n = 1, 2, 3 \ldots \infty$$

b) From the result of part (a), derive an expression for the number of allowed energies in the range from 0 to $E$.

$$n = \frac{\sqrt{2mL}}{\hbar \pi} \sqrt{E} = \text{# level, from 0 to } E$$

c) Now, derive an expression for the fermi energy of the gas at zero temperature in terms of $N/L$, the number of fermions per unit length.

"Two spin $\uparrow$ fermions can go into each level so, filling all states from 0 to $E_F$"

$$N = 2 \frac{\sqrt{2mL}}{\hbar \pi} \sqrt{E_F}$$

$$E_F = \frac{\hbar^2 \pi^2}{8m} \left( \frac{N}{L} \right)^2$$
13) (12 pts) Assume, as a crude approximation, that the two electrons in a helium atom do not interact with each other. If the functions \( \psi_{n,\ell,m_{\ell}}(r,\theta,\phi) \) are the wave functions of a single electron in a Coulomb potential with \( Z=2 \) and \( m_{s} = \pm \frac{1}{2} \) are the spin states of the \( i \)th electron, then

a) write down a wave function for the ground state of the 2-electron system,

\[
\Psi = \frac{1}{\sqrt{2}} \psi_{100}(\vec{r}_1) \psi_{100}(\vec{r}_2) \left[ |m_{s1}=\frac{1}{2}, m_{s2}=-\frac{1}{2} \rangle - |m_{s1}=-\frac{1}{2}, m_{s2}=\frac{1}{2} \rangle \right]
\]

b) and write down a wave function for the first excited state of the 2-electron system with parallel electron spins.

\[
\Psi = \frac{1}{\sqrt{2}} \left[ \psi_{100}(\vec{r}_1) \psi_{100}(\vec{r}_2) - \psi_{100}(\vec{r}_2) \psi_{100}(\vec{r}_1) \right] \chi_{\text{spin}}
\]

where

\[
\chi_{\text{spin}} = \begin{cases} 
|l_{s1}=\frac{1}{2}, l_{s2}=\frac{1}{2}, m_{s1}=m_{s2} \rangle & \text{Any of these is correct}
\end{cases}
\]

14) (8 pts) In the Fermi-Dirac distribution function

\[
\bar{n}_E = \frac{1}{e^{E/kT} + 1}
\]

a) what is the range of validity for \( C \)?

\( C \geq 0 \), so \( 0 \leq \bar{n}_E \leq 1 \) for all \( E, T \)

b) List two physical quantities that are properties of any particular system in thermal equilibrium upon which the value of \( C \) for that system will typically depend.

\begin{align*}
\text{Temperature} & \quad \text{and} \quad \text{total number of particles per unit volume, or } T \text{ and } E_F, \text{ the Fermi energy.}
\end{align*}

c) What is the value of \( C \) for the special case of a gas of massless fermions (such as neutrinos)?

\( C = 1 \) as proved in the text and lecture.