Homework Assignment #6
Due Wednesday, November 16 by 4:30 pm

Please note: for these problems I do not want to see integrals done by Mathematica or any other computer program. In particular, all the integrals needed for the hydrogen atom transitions are either very simple trigonometric integrals or are exponential definite integrals, for which the needed formulas may be found on the inside of the back cover of the textbook.

1. Consider a system of two spins (s = \tfrac{1}{2}) that interact by the time-independent Hamiltonian

\[ H = \frac{4\hbar}{\varepsilon} \mathbf{S}_1 \cdot \mathbf{S}_2 \]

a. Solve the time-independent Schrödinger Equation for the eigenstates and eigenvalues.
b. If the system starts out at t = 0 in the initial state ↑↓, what is the probability that it is found in this state at a later time t? This should be an exact solution, not perturbation theory.
c. For the same initial condition, what is the probability that it is found in the state ↓↑ at a later time t?
d. Now treat this Hamiltonian as a perturbation (with \( H_0 = 0 \)) that is turned on at time \( t = 0 \). Using first-order time-dependent perturbation theory (i.e. Section 9.1.2 in Griffiths), calculate the two time-dependent probabilities as found exactly in (b) and (c). Show that the results are consistent if the parameter \( \varepsilon \) is small.

2. Consider the transition of atoms from an excited state to the ground state. Find the frequency in Hz for which the transition rate from spontaneous emission will be equal to that from thermally stimulated emission at room temperature (300 K). In the latter case assume a black body thermal spectrum for the radiation that stimulates the transition. Which type of transition dominates at optical frequencies?

3. A one-dimensional harmonic oscillator is in its ground state for \( t < 0 \). For \( t \geq 0 \) it is subjected to a time-dependent but spatially uniform force (not potential!) in the +x direction:

\[ F(t) = F_0 e^{-t/\tau}. \]

Using time-dependent perturbation theory to first order, obtain the probability of finding the oscillator in its first excited state for \( t > 0 \). What happens in the limit of very long time? Does it make sense to you? Why?
4. Consider the following dipole matrix elements for hydrogen atom transitions:
   a. Show that \( \langle \psi_{100} | \vec{r} | \psi_{200} \rangle = 0 \), so there are no electric dipole transitions between
      those two states. (In fact there are no transitions for any multipole, period. See
      Problem 9.21.)
   b. Calculate \( \langle \psi_{100} | \vec{r} | \psi_{210} \rangle \) and from the result predict the lifetime of hydrogen in the
      210 excited state.
5. Calculate the lifetime of the hydrogen \( \psi_{300} \) state. See Figure 9.6 for the dominant transitions
   by which it spontaneously decays. Note, however, that the figure does not show the different
   values of \( m_l \) possible in the final state. All have to be considered, as their respective rates
   add together.
6. Widths of spectral lines from a hot atomic gas. The electric field from radiation
   corresponding to a spectral line of angular frequency \( \omega_0 \) will vary with time (for \( t > 0 \)) as
   \( \exp \left( -\frac{1}{2} At + i \omega_0 t \right) \), where \( A \) is the spontaneous decay rate (Einstein’s coefficient). Show
   that the Fourier transform of this function is proportional to
   \[
   \frac{1}{-i(\omega - \omega_0) + \frac{1}{2} A}
   \]
   The spectral intensity will be proportional to the absolute value squared of this expression. Make a plot of it. Also, from the formula find the full width at half maximum of the curve.
   This is the “natural width” of the spectral line. The observed line width will also be
   broadened by Doppler shifts, caused by the random thermal motion of the gas atoms, and
   also by collisions. The collision broadening can be estimated as follows. If there are on
   average \( n \) collisions per second, then we can say that the atom has a chance to vibrate freely
   for only about \( 1/n \) seconds, resulting in an energy spread that can be estimated from the
   energy-time uncertainty relation. A typical atomic lifetime for electric dipole radiation is
   \( 10^{-8} \) s. Calculate and compare the natural, Doppler, and collision broadening of a sodium
   spectral line at 5890 Å formed in a discharge tube at 300°C and at a pressure such that the
   mean free path of a sodium atom is \( 10^{-4} \) cm.