Solve the following three exercises:

1. A cartesian coordinate system with axes $x, y, z$ is rotating relative to an inertial frame with constant angular velocity $\omega$ about the $z$-axis. A particle of mass $m$ moves under a force whose potential is $V(x, y, z)$. Set up the Lagrange equations of motion in the coordinate system $x, y, z$. Show that these equations are the same as those for a particle in a fixed coordinate system acted on by the force $-\nabla V$ and a force derivable from a velocity-dependent potential $U$, and find $U$.

2. A point particle of mass $m$ is constrained to move frictionlessly on the inside surface of a circular wire hoop of radius $r$, uniform density and mass $M$. The hoop is in the $xy$-plane, it can roll on a fixed line (the $x$-axis), but it does not slide, nor can it lose contact with the $x$-axis. The point particle is acted on by gravity exerting a force along the negative $y$-axis. At $t = 0$ suppose the hoop is at rest. At this time the particle is at the top of the hoop, and is given a velocity $v_0$ along the $x$-axis. What is the velocity $v_f$, with respect to the fixed axis, when the particle comes to the bottom of the hoop? Simplify your answer in the limits $m/M \to 0$ and $M/m \to 0$.

3. Assume the Lagrangian for a certain one-dimensional system is given by

$$L = e^{\gamma t} \left( \frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2 \right),$$

where $\gamma, m, k$ are positive constants. Calculate Lagrange’s equations, and give a qualitative description of the particle motion for different values of the constants. Suppose a point transformation is made to another generalized coordinate $S$, given by

$$S = \exp \left( \frac{\gamma t}{2} \right) q.$$

What is the Lagrangian in terms of $S$? Find Lagrange’s equation, and describe the relationship between the motion in the two coordinate systems.