Homework Set #4.

Due Date: Monday December 8, 2011

Solve the following three (plus one optional) exercises:

1. Show that the function
   \[ S = \frac{m\omega}{2} (q^2 + \alpha^2) \cot(\omega t) - m\omega q \alpha \csc(\omega t) \]
   is a solution of the Hamilton-Jacobi equation for Hamilton’s principal function for the linear harmonic oscillator with
   \[ H = \frac{p^2 + m^2 \omega^2 q^2}{2m}. \]
   Show that this function generates a correct solution to the motion of the harmonic oscillator.

2. A particle of mass \( m \) moves in one dimension \( q \) in a potential energy field \( V(q) \) and is retarded by a damping force \(-2m\gamma \dot{q}\) proportional to its velocity.
   (a) Show that the equation of motion can be obtained from the Lagrangian
   \[ L = \exp(2\gamma t) \left( \frac{1}{2}mq^2 - V(q) \right) \]
   and that the Hamiltonian is
   \[ H = \frac{p^2 \exp(-2\gamma t)}{2m} + V(q) \exp(2\gamma t). \]
   (b) For the generating function
   \[ F_2(q, P, t) = \exp(\gamma t)qP \]
   find the transformed Hamiltonian \( K(Q, P, t) \). For an oscillator potential
   \[ V(q) = \frac{1}{2}m\omega^2 q^2 \]
   show that the transformed Hamiltonian yields the constant of motion
   \[ K = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 Q^2 + \gamma QP. \]
(c) Obtain the solution \( q(t) \) for the damped oscillator from the constant of motion in (b) in the under-damped case \( \gamma < \omega \).

You may need the integral

\[
\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x.
\]

3. Consider the logistic equation, defined by the expression

\[ x_{n+1} = ax_n(1-x_n), \quad 0 < x < 1. \]

Show that, to first order in the small parameter \( \delta \), if

\[ x_n = \frac{a-1}{a} \pm \delta \]

then

\[ x_{n+1} = \frac{a-1}{a} \pm \delta(2-a). \]

Infer that the stability range for the fixed point of the logistic equation is \( 1 < a < 3 \).

4. [An optional problem for extra points and extra fun: the Duffing equation attractor].

Solve numerically the Duffing equation:

\[
\frac{d^2 x}{dt^2} = -\delta \frac{dx}{dt} + x - x^3 + \gamma \cos (\omega t),
\]

with \( \delta = 0.20, \gamma = 0.30 \) and \( \omega = 1.0 \).

Solving the equation, you will find a path \( x(t) \): plot several such paths for different initial conditions \( x(t=0) \) and \( x'(t=0) \).

Now, you can obtain the system’s strange attractor by plotting the two-dimensional discrete-time data sample

\[ \{x(nT), \frac{dx}{dt}(nT)\} \]

with \( n = 0, 1, 2, ... \) and \( T = 2\pi/\omega \).

To get a sense of what you should obtain, see the animation at:

http://brain.cc.kogakuin.ac.jp/~kanamaru/Chaos/e/Animation/duffing.html