The cycloid is symmetric about the $y$ axis, so we will only consider the half along the positive $x$ axis. We would like to derive a relation between the initial $y$ coordinate of the particle, $y_0$, and the initial distance along the cycloid from the origin to the particle. We have

$$ds = dy \sqrt{1 + \left(\frac{dx}{dy}\right)^2}$$

$$= dy \sqrt{1 + \left(\frac{dx}{dy}\right)^2}$$

$$= dy \sqrt{1 + \left(\frac{a}{a \sqrt{(2ay - y^2)}/a^2} + \frac{2a - 2y}{2\sqrt{2ay - y^2}}\right)^2}$$

$$= dy \sqrt{1 + \left(\frac{2a - y}{y}\right)^2}$$

$$= dy \sqrt{\frac{2a}{y}}.$$

Note that the range of the half of the cycloid we are considering is $y \in [0, 2a]$, so the integrand blows up as $y \to 0$. However, the integral converges:

$$s(y_0) = \sqrt{2a} \int_0^{y_0} dy \frac{1}{\sqrt{y}} = 2\sqrt{2ay} \bigg|_{y_0}^{y_0} = 2\sqrt{2ay_0}.$$

(Note that for $y = 2a$, this becomes $4a$, a shockingly simple expression.) Now we can write the Lagrangian using the generalized coordinates $s$ and $\dot{s}$:

$$L = \frac{1}{2} m \dot{s}^2 - mg(s) = \frac{1}{2} m \dot{s}^2 - \frac{1}{8a} mgs^2.$$

This is just the Lagrangian for a simple harmonic oscillator with spring constant $g/4a$. Since the period of a SHO does not depend on the starting amplitude, we can similarly conclude that the time the particle takes to reach the origin is independent of $s(y_0)$. To prove this explicitly, we consider the equations of motion:

$$\frac{\partial L}{\partial s} = -\frac{gm}{4a} s = \frac{d}{dt} \frac{\partial L}{\partial \dot{s}} = m\ddot{s} \implies \ddot{s} = -\frac{g}{4a} s.$$

With our initial conditions, this implies that

$$s(t = 0) = 2\sqrt{2ay_0} \cos \left(\sqrt{\frac{g}{4a}} t\right).$$

The time it takes for the particle to reach the origin is solved by solving $s(t_o) = 0$ for $t_o$:

$$0 = 2\sqrt{2ay_0} \cos \left(\sqrt{\frac{g}{4a}} t_o\right) \implies t_o = \pi \sqrt{\frac{a}{g}},$$
as expected. □

2:

Let $x$ be the $x$ coordinate of the hoop, $\theta$ be the angle through which it has rotated, and $\phi$ be the angle between the $-y$ axis and the radial line from the center of the hoop to the particle. (Note that the configuration manifold for this problem is thus $\mathbb{R} \times \mathbb{S}^1$, since the only non-redundant coordinate for the hoop is $x$ and the only one for the particle is $\phi$). Since the hoop rolls without slipping, $x = r \theta$. Let $V_f$ be equal to $\dot{x}$ when the particle comes to the bottom of the hoop.

Assume that $v_0$ is large enough for the particle to stay in contact with the hoop. Then with the above coordinates, we can use conservation of momentum in the $x$ direction and conservation of energy to solve for $v_f$. These two conditions give us

$$mv_0 = mv_f + MV_f$$
$$\frac{1}{2}mv_0^2 + mgr = \frac{1}{2}mv_f^2 - mgr + MV_f^2,$$

where we used the facts that $\dot{\theta} = \frac{\dot{x}}{r}$ and that the moment of inertia for the hoop is $Mr^2$. The first equation can be rearranged to give

$$V_f^2 = \frac{m^2}{M^2}(v_0 - v_f)^2.$$

Substituting into the second obtains

$$\frac{1}{2}mv_0^2 + 2mgr = \frac{1}{2}mv_f^2 + \frac{m^2}{M}(v_0^2 - 2v_0v_f + v_f^2)$$
$$\implies 0 = (2m + M)mv_f^2 - 4m^2v_0v_f + (2m - M)mv_0^2 - 4Mmgr.$$

Using the quadratic formula, we can solve for $v_f$:

$$v_f = \frac{4m^2v_0 - \sqrt{16m^4v_0^2 - 4m(2m + M)(2m - M)mv_0^2 - 4Mmgr}}{2m(2m + M)}$$
$$= \frac{4m^2v_0 - \sqrt{16m^4v_0^2 - 4m^2(4m^2 - M^2)v_0^2 + 16m^2M(2m + M)gr}}{2m(2m + M)}$$
$$= \frac{2mv_0 - \sqrt{M^2v_0^2 + 8Mmgr + 4M^2gr}}{2m + M}$$
$$= \frac{2v_0m/M - \sqrt{m/M(v_0^2M/m + 8 + 4grM/m)}}{2m/M + 1}$$

In the two limits we are asked to consider, this becomes

$$\lim_{m/M \to 0} v_f = \sqrt{m/M(v_0^2M/m + 4grM/m)} = -\sqrt{v_0^2 + 4gr}$$
$$\lim_{M/m \to 0} v_f = \frac{2v_0m/M - \sqrt{8m/M}}{2m/M} = v_0 - \sqrt{\frac{8M}{m}} = v_0.$$

The first limit is what we would have found if the hoop were stationary. The second shows that very little momentum will be transferred to the hoop if it is much lighter than the particle, which agrees with our intuition. □
(By considering what we expect \( v_f \) to be in the limit \( m \ll M \), we can conclude that the negative root is the correct one to take in the quadratic formula. I suspect there is a more satisfactory explanation for why we discard the positive root.)

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**3:**

**a.** Let \( \alpha \) be the angle between rod 1 and vertically downwards, and \( \beta \) be the angle between rod 2 and vertically downwards. \( \alpha \) and \( \beta \) can be taken as our generalized coordinates, in which case we see that the configuration space \( Q \) is just \( S^1 \times S^1 \equiv T^2 \), the 2-torus. The tangent space at any point in an \( n \) dimensional manifold is isomorphic to \( \mathbb{R}^n \), so the tangent space to a point of \( Q \) is isomorphic to \( \mathbb{R}^n \). The tangent bundle is the disjoint union

\[
TQ = \bigsqcup_{q \in Q} T_q T^2,
\]

along with the projection \( \pi : TQ \to Q \) as well as the standard topology and smooth structure. □

**b.** We will use the coordinates mentioned above, \((q; v) = (\alpha, \beta; \dot{\alpha}, \dot{\beta})\). The cartesian coordinates of each mass can be written as follows:

\[
(x_1, y_1) = (L_1 \sin \alpha, -L_1 \cos \alpha)
\]

\[
v_1 = (\dot{\alpha} L_1 \cos \alpha, \dot{\alpha} L_1 \sin \alpha)
\]

\[
\implies v_1 = L_1^2 \dot{\alpha}^2
\]

\[
(x_2, y_2) = (x_1 + L_2 \sin \beta, y_1 + L_2 \cos \beta) = (L_2 \sin \beta + L_1 \sin \alpha, -L_2 \cos \beta - L_1 \cos \alpha)
\]

\[
v_2 = (\dot{\beta} L_2 \cos \beta + \dot{\alpha} L_1 \cos \alpha, \dot{\beta} L_2 \sin \beta + \dot{\alpha} L_1 \sin \alpha)
\]

\[
\implies v_2 = \dot{\beta}^2 L_2^2 + \dot{\alpha}^2 L_1^2 + \dot{\alpha} \dot{\beta} L_1 L_2 \sin(\alpha + \beta).
\]

Then the Lagrangian is

\[
L = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 - m_1 g y_1 - m_2 g y_2
\]

\[
= \frac{1}{2} (m_1 + m_2) L_1^2 \dot{\alpha}^2 + \frac{1}{2} m_2 (\dot{\beta}^2 L_2^2 + \dot{\alpha} \dot{\beta} L_1 L_2 \sin(\alpha + \beta)) + g(m_1 + m_2) L_1 \sin \alpha + g m_2 L_2 \cos \beta. \quad \square
\]

**c.** The EL equations for \( \alpha \) give us

\[
\frac{\partial L}{\partial \dot{\alpha}} = \frac{d}{dt} \frac{\partial L}{\partial \dot{\alpha}}
\]

\[
= \frac{1}{2} m_2 \ddot{\alpha} L_1 L_2 \cos(\alpha + \beta) - m_1 g L_1 \sin \alpha - m_2 g L_1 \sin \alpha
\]

\[
= (m_1 + m_2) L_1^2 \ddot{\alpha} + \frac{1}{2} m_2 \ddot{\beta} L_1 L_2 \sin(\alpha + \beta) + \frac{1}{2} m_2 L_1 \ddot{\beta} (\dot{\alpha} + \dot{\beta}) \cos(\alpha + \beta)
\]

\[
\implies m_2 L_1 L_2 \ddot{\beta} \sin(\alpha + \beta) + m_2 L_1 L_2 \ddot{\beta} \cos(\alpha + \beta) + 2(m_1 + m_2) L_1^2 \ddot{\alpha} + 2(m_1 + m_2) L_1 \sin \alpha = 0.
\]

For \( \beta \), the EL equations become

\[
\frac{\partial L}{\partial \dot{\beta}} = \frac{d}{dt} \frac{\partial L}{\partial \dot{\beta}}
\]

\[
= \frac{1}{2} m_2 L_1 L_2 \dot{\alpha} \dot{\beta} \cos(\alpha + \beta) - g m_2 L_2 \sin \beta
\]

\[
= m_2 L_2 \ddot{\beta} + \frac{1}{2} m_2 L_1 L_2 \ddot{\alpha} \sin(\alpha + \beta) + \frac{1}{2} m_2 L_1 L_2 \ddot{\alpha} (\dot{\alpha} + \dot{\beta}) \sin(\alpha + \beta)
\]

\[
\implies m_2 L_1 L_2 \ddot{\alpha} \sin(\alpha + \beta) + m_2 L_1 L_2 \ddot{\alpha} \sin(\alpha + \beta) + 2 m_2 L_2 \ddot{\beta} + g m_2 L_2 \sin \beta = 0. \quad \square
\]
4: Rotating coordinate system

The transformation taking us from the rotating coordinate system \( O \) and the inertial one \( \overline{O} \) is some kind of rotation matrix. Consider a particle starting at position \((x, y) = (x_0, 0)\). Then by geometry, \((\bar{x}, \bar{y}) = (x_0 \cos \omega t, x_0 \sin \omega t)\). Similarly, a particle at position \((x, y) = (0, y_0)\) in the rotating frame has coordinates \((\bar{x}, \bar{y}) = (-y_0 \sin \omega t, y_0 \cos \omega t)\) in the inertial frame. Since the coordinate transformation is a rotation and is therefore linear, the full transformation is

\[
R(t) \mathbf{x} = \mathbf{x} \rightarrow \begin{pmatrix} \cos \omega t & -\sin \omega t \\ \sin \omega t & \cos \omega t \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}.
\]

Velocities in the inertial coordinate system transform as

\[
\begin{align*}
\dot{\bar{x}} &= (\dot{x} - \omega y) \cos \omega t - (\dot{y} + \omega x) \sin \omega t \\
\dot{\bar{y}} &= (\dot{x} - \omega y) \sin \omega t + (\dot{y} + \omega x) \cos \omega t \\
\end{align*}
\]

In the inertial frame,

\[
L(\bar{x}, \bar{y}, \bar{z}, \dot{\bar{x}}, \dot{\bar{y}}, \dot{\bar{z}}) = \frac{1}{2} m(\dot{\bar{x}}^2 + \dot{\bar{y}}^2 + \dot{\bar{z}}^2) - V(\bar{x}, \bar{y}, \bar{z}).
\]

Under the coordinate transformation to the rotating frame, the Lagrangian becomes

\[
L(x, y, \dot{x}, \dot{y}) = \frac{1}{2} m[(\dot{x} - \omega y)^2 + (\dot{y} + \omega x)^2 + \dot{z}^2] - V(x, y, z)
\]

\[
= \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x, y, z) + \frac{1}{2} m\omega^2(x^2 + y^2) + m\omega(x\dot{y} - y\dot{x}).
\]

This is the sum of the Lagrangian for a particle moving under potential \( V \) in the \( O \) frame and a velocity dependent potential \( U = -\frac{1}{2} m\omega^2(x^2 + y^2) + m\omega(y\dot{x} - x\dot{y}) \). The generalized force associated with \( U \) is

\[
\begin{align*}
Q_x &= -\frac{\partial U}{\partial x} + \frac{d}{dt} \frac{\partial U}{\partial \dot{x}} = m\omega^2 x + m\omega y \\
Q_y &= -\frac{\partial U}{\partial y} + \frac{d}{dt} \frac{\partial U}{\partial \dot{y}} = m\omega^2 y - m\omega x \\
\implies Q &= m\omega^2 \rho\dot{\rho} + m\omega(y\dot{x} - x\dot{y}) \quad \square
\end{align*}
\]