1:

a. Even though the question doesn’t specify the shape of the dust cloud, I am going to make
the assumption that the dust cloud planet-sun system a sphere centered at the sun whose radius
is very large. If the dust cloud were truly infinite, it would have additional symmetries, and the
answer to the question would not be the one we are supposed to obtain.

We can break the distribution of dust into an infinite number of infinitesimally thin spherical
shells. Let the distance between the sun and planet be \( r \). By Newton’s shell theorem, we can ignore
the gravitational force due to dust outside of this shell. The force exerted by the dust on the planet
is therefore

\[
F' = \frac{Gm}{r^2} \cdot \frac{4}{3} \pi r^3 \rho \hat{r} = -\frac{4 \pi G \rho}{3} m \frac{r}{r}.
\]

□

b. The attractive forces from the sun and dust provide the centripetal force which keeps the
planet in its circular orbit:

\[
\frac{mv^2}{r_0} = \frac{L^2}{mr^2} = \frac{GMm}{r_0^2} + mk r_0.
\]

This can be solved to find \( r_0 \).

\( \square \)

c. Consider what happens in the sun’s rest frame. Such a coordinate transformation is legiti-
mate since the sun’s frame is inertial. We assume that the sun has been traveling through the dust
cloud for a long time, so the dust comes towards the sun from \( r = \infty \) with velocity \( V \). The dust
particles therefore have hyperbolic orbits about the sun of the form

\[
r = \frac{l^2}{GMm^2 [1 + e \cos(\theta - \theta_0)]}
\]

\[
e \equiv \sqrt{1 + \frac{2El^2}{G^2M^2m^3}} = \sqrt{1 + \frac{V^4s^2}{G^2M^2}},
\]

where \( \theta_0 \) is the angular coordinate of the dust particle at its closest approach to the sun. The
closest the particle gets to the sun is therefore

\[
r_m = \frac{l^2}{GMm^2(1 + e)} = \frac{V^2s^2}{GM \left[ 1 + \sqrt{1 + \frac{V^4s^2}{G^2M^2}} \right]}.
\]

If \( r_m \leq R \), the particle will be absorbed by the sun. This occurs at an impact parameter defined
by the following expression:

\[
\sqrt{1 + \frac{V^4s^2}{G^2M^2}} = \frac{V^2s^2}{GM R} - 1
\]

\[
\frac{V^4s^2}{G^2M^2} = \frac{V^4s^4}{G^2M^2R^2} - \frac{2V^2s^2}{GM R}
\]

\[
s_c = \sqrt{R^2 + \frac{2GM R}{V^2}}.
\]
This “critical impact parameter” we’ve derived blows up when $V = 0$, which makes sense: it shows that every particle will be attracted to the sun. The part of the dust cloud that will hit the sun in an interval $dt$ is therefore initially contained in a cylinder of radius $s_c$ and thickness $dx = Vdt$ located at $r = \infty$. Thus

$$\frac{dm}{dt} = \pi s_c^2 V \rho = \pi R^2 V \rho + \frac{2\pi GM \rho}{V}.$$ 

This is the geometric cross section of the sun plus a gravitational enhancement. Note that this result does not apply when $V = 0$.

**d.** This can be treated as a 1D problem. By Newton’s second law,

$$\frac{dp}{dt} = F = \frac{dm}{dt} v(t) + \frac{dv}{dt} m(t).$$

If we are to assume that $\dot{V} = 0$ as we did in part c, the last term vanishes, and $v(t)$ becomes $V$. (Note that with this assumption, the drag force is really the force required to keep the sun moving with velocity $V$ at the instant before it has accreted any mass.) With the additional assumption that the interactions between dust particles can be ignored, $\dot{m}$ becomes what we found in part c, and the drag force is

$$F = \pi R^2 \rho V^2 + 2\pi GM \rho.$$ 

The drag force points in the opposite direction of the sun’s velocity. Note that this result does not apply when $V = 0$ either.

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**2:**

Conservation of momentum gives us

$$m_1 v_0 = m_1 v_1 \cos \vartheta + m_2 v_2 \cos \alpha, \quad m_1 v_1 \sin \vartheta = m_2 v_2 \sin \alpha,$$

where $\alpha$ is the angle between $v_2$ and the positive $x$ axis. Rearranging the first equation, squaring both equations, adding and multiplying by 1/2 yields

$$\frac{1}{2} \left( m_1^2 v_0^2 - 2m_1 m_2 v_0 v_1 \cos \vartheta + m_2^2 v_1^2 \cos^2 \vartheta = m_2^2 v_2^2 \cos^2 \alpha \right) + \frac{1}{2} \left( m_2^2 v_2^2 \sin^2 \vartheta = m_2^2 v_2^2 \sin^2 \alpha \right)$$

$$\frac{1}{2} m_1^2 v_1^2 + \frac{1}{2} m_2^2 v_2^2 - m_1 m_2 v_0 v_1 \cos \vartheta = \frac{1}{2} m_2^2 v_2^2.$$ 

The energies are defined by

$$E_0 = \frac{1}{2} m_1 v_0^2, \quad E_1 = \frac{1}{2} m_1 v_1^2,$$

so we can substitute for the velocities:

$$\frac{1}{2} m_2^2 v_2^2 = m_1 E_1 + m_1 E_0 - 2m_1 \sqrt{E_0 E_1} \cos \vartheta$$

$$\implies \cos \vartheta = \frac{1}{2} \sqrt{\frac{E_1}{E_0}} + \frac{1}{2} \sqrt{\frac{E_0}{E_1}} - \frac{m_2^2 v_2^2}{4m_1 \sqrt{E_0 E_1}}.$$
The $Q$ value is defined as

\[ E_0 + Q = E_1 + \frac{1}{2}m_2v_2^2 \implies \frac{1}{2}m_2v_2^2 = m_2E_0 + m_2Q - m_2E_1. \]

Substituting obtains

\[
cos \vartheta = \frac{m_1 + m_2}{2m_1} \sqrt{\frac{E_1}{E_0}} + \frac{m_1}{2m_1} \sqrt{\frac{E_0}{E_1}} - \frac{m_2 - m_1}{2m_1} \sqrt{\frac{E_0}{E_1}} - \frac{m_2Q}{2m_1 \sqrt{E_0E_1}}. \quad \square
\]

3:

Recall that

\[
\Theta(s) = \pi - 2 \int_0^{u_m} \frac{s \ du}{\sqrt{1 - \frac{V(u)}{E_0} - s^2u^2}}
\]

where $u_m \equiv 1/r_m$ is the distance of closest approach. The potential that results in the given force law is

\[ V(r) = \frac{k}{2r^2}. \]

Let $v_m$ be the velocity of the particle at its closest approach. Conservation of energy and angular momentum give $r_m$:

\[
l = s\sqrt{2mE_0} = mv_mr_m \implies \frac{1}{2}mv_m^2 = \frac{E_0s^2}{r_m^2}
\]

\[ E_0 = \frac{1}{2}mv_m^2 + \frac{k}{2r_m^2} = \frac{E_0s^2}{r_m^2} + \frac{k}{2r_m^2}
\]

\[ \implies r_m = \sqrt{s^2 + \frac{k}{2E_0}}.
\]

With this information, $\Theta$ can be evaluated

\[
\Theta(s) = \pi - 2s \int_0^{u_m} \frac{du}{\sqrt{1 - \left(\frac{k}{2E_0} + s^2\right)u^2}}
\]

\[ = \pi - 2s \sin^{-1}\left(\frac{\frac{1}{r_m} \sqrt{\frac{k}{2E_0} + s^2}}{\sqrt{\frac{k}{2E_0} + s^2}}\right)
\]

\[ = \pi - 2s \sin^{-1}\left(\frac{1}{r_m}\right)\]

\[ x = 1 - \frac{s}{\sqrt{\frac{k}{2E_0} + s^2}}.
\]
Rearranging yields

\[ s^2 = \frac{k}{2E_0} \cdot \frac{(1-x)^2}{x(2-x)} \]

\[ \Rightarrow \frac{ds^2}{dx} = \frac{k}{E_0} \cdot \frac{1-x}{(2-x)^2 x^2}. \]

The cross section is

\[ \Rightarrow \sigma(\Theta) = \frac{s}{\pi \sin \Theta} \left| \frac{ds}{dx} \right| \]

\[ \frac{ds}{dx} = \frac{ds}{ds^2} \frac{ds^2}{dx} = \frac{1}{2s} \frac{ds^2}{dx} \]

\[ \Rightarrow \sigma(\Theta) = \frac{s}{\pi \sin(\pi x)} \cdot \frac{1}{2s} \cdot \frac{k}{E_0} \cdot \frac{1-x}{(2-x)^2 x^2} \]

\[ \Rightarrow \sigma(\Theta) d\Theta = \frac{k}{2E_0} \cdot \frac{1-x}{(2-x)^2 x^2 \sin(\pi x)} dx. \]