The logarithmic derivative of the Gamma function

In this note, I will sketch some of the main properties of the logarithmic derivative* of the Gamma function. The formal definition is given by:

$$\psi(x) \equiv \frac{d}{dx} \ln \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)} \,,$$

where $\Gamma'(x)$ is the ordinary derivative of $\Gamma(x)$ with respect to x. We also define:

$$\lim_{x \to 1} \Gamma'(x) = -\gamma \,,$$

where $\gamma \simeq 0.5772156649 \cdots$ is known as Euler's constant. It is not known whether γ is a rational or irrational number (although there is strong suspicion for the latter). By definition of $\psi(x)$, we see [since $\Gamma(1) = 1$] that:

$$\psi(1) = -\gamma \,.$$

Using $\Gamma(x+1) = x\Gamma(x)$, we can differentiate this equation to derive a fundamental property of $\psi(x)$.

$$\Gamma'(x+1) = \Gamma(x) + x\Gamma'(x) \,,$$

$$\frac{\Gamma'(x+1)}{\Gamma(x)} = 1 + x\frac{\Gamma'(x)}{\Gamma(x)} \,.$$

Finally, writing $\Gamma(x) = \Gamma(x+1)/x$ on the left hand side above, and then dividing through by x, we find:

$$\boxed{\psi(x+1) = \frac{1}{x} + \psi(x) \,.} \tag{1}$$

Consider the case of $x = n = 0, 1, 2, \ldots$. Then, using eq. (1)

$$\psi(n+1) = \frac{1}{n} + \psi(n) = \frac{1}{n} + \frac{1}{n-1} + \psi(n-1) = \cdots \,,$$

until we reach $\psi(1) = -\gamma$. The end result is:

$$\boxed{\psi(n+1) = -\gamma + \sum_{k=1}^{n} \frac{1}{k} \,, \qquad n = 0, 1, 2, \ldots} \tag{2}$$

*The *logarithmic derivative* of a function is defined as the derivative of the logarithm of the function.

Eq. (2) provides an interesting connection between the logarithmic derivative of the Gamma function and the finite harmonic series.

We next examine the asymptotic behavior of $\psi(x)$ as $x \to \infty$. This is easily accomplished by making use of Stirling's formula:

$$\ln \Gamma(x+1) = (x + \tfrac{1}{2}) \ln x - x + \tfrac{1}{2} \ln 2\pi + \mathcal{O}(x^{-1}) \,, \qquad \text{as } x \to \infty \,.$$

Differentiating this formula yields the large x asymptotic behavior of $\psi(x+1)$:

$$\psi(x+1) = \ln x + \frac{1}{2x} + \mathcal{O}(x^{-2}) \,, \qquad \text{as } x \to \infty \,. \tag{3}$$

In particular, it follows that for an integer n,

$$\lim_{n\to\infty} \psi(n+1) - \ln n = 0 \,.$$

If we use result for $\psi(n+1)$ given in eq. (2), we conclude that:

$$\gamma = \lim_{n\to\infty} \left[\sum_{k=1}^{n} \frac{1}{k} - \ln n \right] \,.$$

This is a remarkable formula, which often serves as the definition of Euler's constant. (Textbooks that adopt this definition must spend some time proving that this limit exists and is finite.) The above results also imply that:

$$\sum_{k=1}^{n} \frac{1}{k} = \ln n + \gamma + \frac{1}{2n} + \mathcal{O}\left(\frac{1}{n^2}\right) \,.$$

This result provides the start of an asymptotic expansion for the finite harmonic sum as $n \to \infty$. Moreover, it tells us that the infinite harmonic sum diverges logarithmically (which explains the slow growth of the corresponding finite sums).

So far, we only have an explicit formula for $\psi(x)$ when x is a positive integer [eq. (2)]. We can derive a more general result as follows. In the same way that we derived eq. (2), we may use eq. (1) to obtain:

$$\begin{aligned} \psi(x+n) &= \frac{1}{x+n-1} + \frac{1}{x+n-2} + \cdots + \frac{1}{x} + \psi(x) \\ &= \sum_{k=0}^{n-1} \frac{1}{x+k} + \psi(x) \,, \end{aligned} \tag{4}$$

where n is a positive integer and x is arbitrary. Subtracting eq. (2) from eq. (4) yields:

$$\psi(x+n) - \psi(n+1) = \sum_{k=0}^{n-1} \left(\frac{1}{x+k} - \frac{1}{k+1} \right) + \gamma + \psi(x) \,. \tag{5}$$

Consider the $n \to \infty$ limit of eq. (5). Using eq. (3), it follows that

$$\lim_{n\to\infty} \psi(x+n) - \psi(n+1) = \mathcal{O}(n^{-1}) \to 0\,.$$

Hence, we conclude that:

$$\boxed{\psi(x) = -\gamma - \sum_{k=0}^{\infty}\left(\frac{1}{x+k} - \frac{1}{k+1}\right)\,.} \qquad (6)$$

You should check that if x is a positive integer, then eq. (6) reduces to eq. (2).

Starting from eq. (6), we can differentiate one more time with respect to x to obtain the second derivative of $\ln\Gamma(x)$. The result is†

$$\frac{d^2\ln\Gamma(x)}{dx^2} = \frac{d\psi}{dx} = \sum_{k=0}^{\infty}\frac{1}{(x+k)^2}\,. \qquad (7)$$

We can express this second derivative in terms of the Gamma function and its derivatives:

$$\frac{d^2\ln\Gamma(x)}{dx^2} = \frac{d}{dx}\frac{\Gamma'(x)}{\Gamma(x)} = \frac{\Gamma''(x)}{\Gamma(x)} - \left[\frac{\Gamma'(x)}{\Gamma(x)}\right]^2\,.$$

If we set $x = 1$ and use $\Gamma(1) = 1$ and $\Gamma'(1) = -\gamma$, then eq. (7) yields

$$\Gamma''(1) - \gamma^2 = \sum_{k=0}^{\infty}\frac{1}{(k+1)^2} = \frac{\pi^2}{6}\,,$$

after recognizing the value of the infinite sum in the last equation. Thus,

$$\Gamma''(1) = \gamma^2 + \frac{\pi^2}{6}\,.$$

With this information, we now know the first three terms of the Taylor series of $\Gamma(1+x)$, expanded about $x = 0$:

$$\Gamma(1+x) = 1 - \gamma x + \tfrac{1}{2}\left(\gamma^2 + \frac{\pi^2}{6}\right) + \mathcal{O}(x^3)\,. \qquad (8)$$

This particular expansion plays an important role in some quantum field theory computations.

†Eq. (7) implies that the second derivative of $\ln\Gamma(x)$ is positive for all real and positive x. That is, the Gamma function is *log-convex* for positive values of x. I mention this fact because one can prove that given a function $\Gamma(x)$ that satisfies $\Gamma(1) = 1$, $x\Gamma(x) = \Gamma(x+1)$ and is log-convex for $0 < x < \infty$, then this function is *uniquely* defined. Moreover, there is a unique extension of this function to the negative real axis (and more generally to the full complex plane).

For completeness, let us consider the nth logarithmic derivative of the Gamma function:

$$\psi^{(n-1)}(x) \equiv \frac{d^n}{dx^n} \ln \Gamma(x) \,,$$

for $n = 1, 2, 3, \ldots$, with $\psi^{(0)}(x) \equiv \psi(x)$. Taking further derivatives of eq. (7) then yields:

$$\psi^{(n)}(x) = \frac{d^{n+1} \ln \Gamma(x)}{dx^{n+1}} = (-1)^{n+1}\, n! \sum_{k=0}^{\infty} \frac{1}{(x+k)^{n+1}} \,, \qquad n = 1, 2, 3, \ldots$$

If we set $x = 1$, then using the definition of the Riemann zeta function:

$$\zeta(n) = \sum_{k=0}^{\infty} \frac{1}{(k+1)^n} \,, \qquad n = 2, 3, 4, \ldots \,,$$

it follows that $\psi^{(n)}(1) = (-1)^{n-1}\, n!\, \zeta(n+1)$ for $n = 1, 2, 3, \ldots$. We now have enough information to write down the complete Maclaurin series expansion for $\ln \Gamma(1+x)$:

$$\ln \Gamma(1+x) = -\gamma x + \sum_{n=2}^{\infty} (-1)^n \frac{x^n}{n} \zeta(n) \,. \tag{9}$$

One can check that this series converges for $-1 < x \leq 1$. Exponentiating both sides of this equation and expanding out the right hand side around $x = 0$ would yield the Maclaurin expansion for $\Gamma(1+x)$. The first three terms of this latter expansion have already been obtained in eq. (8). You should verify this explicitly, noting that $\zeta(2) = \pi^2/6$.

Finally, if we put $x = 1$ in eq. (9) and use $\Gamma(2) = 1$, we can deduce yet another (rather slowly converging) series representation for Euler's constant:

$$\gamma = \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta(n) \,.$$